

**Sixth Term Examination Papers**

**9475**

**MATHEMATICS 3**

Morning

**FRIDAY 19 JUNE 2015**

Time: 3 hours

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Additional Materials: Answer Booklet  
Formulae Booklet

### **INSTRUCTIONS TO CANDIDATES**

**Please read this page carefully, but do not open this question paper until you are told that you may do so.**

Write your name, centre number and candidate number in the spaces on the answer booklet.

Begin each answer on a new page.

Write the numbers of the questions you answer in the order attempted on the front of the answer booklet.

### **INFORMATION FOR CANDIDATES**

Each question is marked out of 20. There is no restriction of choice.

All questions attempted will be marked.

Your final mark will be based on the **six** questions for which you gain the highest marks.

You are advised to concentrate on no more than **six** questions. Little credit will be given for fragmentary answers.

You are provided with a Mathematical Formulae Booklet.

**Calculators are not permitted.**

**Please wait to be told you may begin before turning this page.**

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This question paper consists of 9 printed pages and 3 blank pages.

## Section A: Pure Mathematics

1 (i) Let

$$I_n = \int_0^{\infty} \frac{1}{(1+u^2)^n} du,$$

where  $n$  is a positive integer. Show that

$$I_n - I_{n+1} = \frac{1}{2n} I_n$$

and deduce that

$$I_{n+1} = \frac{(2n)! \pi}{2^{2n+1} (n!)^2}.$$

(ii) Let

$$J = \int_0^{\infty} f((x-x^{-1})^2) dx,$$

where  $f$  is any function for which the integral exists. Show that

$$J = \int_0^{\infty} x^{-2} f((x-x^{-1})^2) dx = \frac{1}{2} \int_0^{\infty} (1+x^{-2}) f((x-x^{-1})^2) dx = \int_0^{\infty} f(u^2) du.$$

(iii) Hence evaluate

$$\int_0^{\infty} \frac{x^{2n-2}}{(x^4-x^2+1)^n} dx,$$

where  $n$  is a positive integer.

**2** If  $s_1, s_2, s_3, \dots$  and  $t_1, t_2, t_3, \dots$  are sequences of positive numbers, we write

$$(s_n) \leq (t_n)$$

to mean

“there exists a positive integer  $m$  such that  $s_n \leq t_n$  whenever  $n \geq m$ ”.

Determine whether each of the following statements is true or false. In the case of a true statement, you should give a proof which includes an explicit determination of an appropriate  $m$ ; in the case of a false statement, you should give a counterexample.

(i)  $(1000n) \leq (n^2)$ .

(ii) If it is not the case that  $(s_n) \leq (t_n)$ , then it is the case that  $(t_n) \leq (s_n)$ .

(iii) If  $(s_n) \leq (t_n)$  and  $(t_n) \leq (u_n)$ , then  $(s_n) \leq (u_n)$ .

(iv)  $(n^2) \leq (2^n)$ .

**3** In this question,  $r$  and  $\theta$  are polar coordinates with  $r \geq 0$  and  $-\pi < \theta \leq \pi$ , and  $a$  and  $b$  are positive constants.

Let  $L$  be a fixed line and let  $A$  be a fixed point not lying on  $L$ . Then the locus of points that are a fixed distance (call it  $d$ ) from  $L$  measured along lines through  $A$  is called a *conchoid of Nicomedes*.

(i) Show that if

$$|r - a \sec \theta| = b, \tag{*}$$

where  $a > b$ , then  $\sec \theta > 0$ . Show that all points with coordinates satisfying (\*) lie on a certain conchoid of Nicomedes (you should identify  $L$ ,  $d$  and  $A$ ). Sketch the locus of these points.

(ii) In the case  $a < b$ , sketch the curve (including the loop for which  $\sec \theta < 0$ ) given by

$$|r - a \sec \theta| = b.$$

Find the area of the loop in the case  $a = 1$  and  $b = 2$ .

[Note:  $\int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C$ .]

- 4 (i) If  $a$ ,  $b$  and  $c$  are all real, show that the equation

$$z^3 + az^2 + bz + c = 0 \quad (*)$$

has at least one real root.

- (ii) Let

$$S_1 = z_1 + z_2 + z_3, \quad S_2 = z_1^2 + z_2^2 + z_3^2, \quad S_3 = z_1^3 + z_2^3 + z_3^3,$$

where  $z_1$ ,  $z_2$  and  $z_3$  are the roots of the equation (\*). Express  $a$  and  $b$  in terms of  $S_1$  and  $S_2$ , and show that

$$6c = -S_1^3 + 3S_1S_2 - 2S_3.$$

- (iii) The six real numbers  $r_k$  and  $\theta_k$  ( $k = 1, 2, 3$ ), where  $r_k > 0$  and  $-\pi < \theta_k < \pi$ , satisfy

$$\sum_{k=1}^3 r_k \sin(\theta_k) = 0, \quad \sum_{k=1}^3 r_k^2 \sin(2\theta_k) = 0, \quad \sum_{k=1}^3 r_k^3 \sin(3\theta_k) = 0.$$

Show that  $\theta_k = 0$  for at least one value of  $k$ .

Show further that if  $\theta_1 = 0$  then  $\theta_2 = -\theta_3$ .

- 5 (i) In the following argument to show that  $\sqrt{2}$  is irrational, give proofs appropriate for steps 3, 5 and 6.

1. Assume that  $\sqrt{2}$  is rational.

2. Define the set  $S$  to be the set of positive integers with the following property:

$$n \text{ is in } S \text{ if and only if } n\sqrt{2} \text{ is an integer.}$$

3. Show that the set  $S$  contains at least one positive integer.

4. Define the integer  $k$  to be the smallest positive integer in  $S$ .

5. Show that  $(\sqrt{2} - 1)k$  is in  $S$ .

6. Show that steps 4 and 5 are contradictory and hence that  $\sqrt{2}$  is irrational.

- (ii) Prove that  $2^{\frac{1}{3}}$  is rational if and only if  $2^{\frac{2}{3}}$  is rational.

Use an argument similar to that of part (i) to prove that  $2^{\frac{1}{3}}$  and  $2^{\frac{2}{3}}$  are irrational.

- 6 (i) Let  $w$  and  $z$  be complex numbers, and let  $u = w + z$  and  $v = w^2 + z^2$ . Prove that  $w$  and  $z$  are real if and only if  $u$  and  $v$  are real and  $u^2 \leq 2v$ .

- (ii) The complex numbers  $u$ ,  $w$  and  $z$  satisfy the equations

$$\begin{aligned}w + z - u &= 0 \\w^2 + z^2 - u^2 &= -\frac{2}{3} \\w^3 + z^3 - \lambda u &= -\lambda\end{aligned}$$

where  $\lambda$  is a positive real number. Show that for all values of  $\lambda$  except one (which you should find) there are three possible values of  $u$ , all real.

Are  $w$  and  $z$  necessarily real? Give a proof or counterexample.

- 7 An operator  $D$  is defined, for any function  $f$ , by

$$Df(x) = x \frac{df(x)}{dx}.$$

The notation  $D^n$  means that  $D$  is applied  $n$  times; for example

$$D^2f(x) = x \frac{d}{dx} \left( x \frac{df(x)}{dx} \right).$$

Show that, for any constant  $a$ ,  $D^2x^a = a^2x^a$ .

- (i) Show that if  $P(x)$  is a polynomial of degree  $r$  (where  $r \geq 1$ ) then, for any positive integer  $n$ ,  $D^n P(x)$  is also a polynomial of degree  $r$ .
- (ii) Show that if  $n$  and  $m$  are positive integers with  $n < m$ , then  $D^n(1-x)^m$  is divisible by  $(1-x)^{m-n}$ .
- (iii) Deduce that, if  $m$  and  $n$  are positive integers with  $n < m$ , then

$$\sum_{r=0}^m (-1)^r \binom{m}{r} r^n = 0.$$

- 8 (i) Show that under the changes of variable  $x = r \cos \theta$  and  $y = r \sin \theta$ , where  $r$  is a function of  $\theta$  with  $r > 0$ , the differential equation

$$(y + x) \frac{dy}{dx} = y - x$$

becomes

$$\frac{dr}{d\theta} + r = 0.$$

Sketch a solution in the  $x$ - $y$  plane.

- (ii) Show that the solutions of

$$(y + x - x(x^2 + y^2)) \frac{dy}{dx} = y - x - y(x^2 + y^2)$$

can be written in the form

$$r^2 = \frac{1}{1 + Ae^{2\theta}}$$

and sketch the different forms of solution that arise according to the value of  $A$ .

## Section B: Mechanics

- 9 A particle  $P$  of mass  $m$  moves on a smooth fixed straight horizontal rail and is attached to a fixed peg  $Q$  by a light elastic string of natural length  $a$  and modulus  $\lambda$ . The peg  $Q$  is a distance  $a$  from the rail. Initially  $P$  is at rest with  $PQ = a$ .

An impulse imparts to  $P$  a speed  $v$  along the rail. Let  $x$  be the displacement at time  $t$  of  $P$  from its initial position. Obtain the equation

$$\dot{x}^2 = v^2 - k^2 \left( \sqrt{x^2 + a^2} - a \right)^2$$

where  $k^2 = \lambda/(ma)$ ,  $k > 0$  and the dot denotes differentiation with respect to  $t$ .

Find, in terms of  $k$ ,  $a$  and  $v$ , the greatest value,  $x_0$ , attained by  $x$ . Find also the acceleration of  $P$  at  $x = x_0$ .

Obtain, in the form of an integral, an expression for the period of the motion. Show that, in the case  $v \ll ka$  (that is,  $v$  is much less than  $ka$ ), this is approximately

$$\sqrt{\frac{32a}{kv}} \int_0^1 \frac{1}{\sqrt{1-u^4}} du.$$

- 10 A light rod of length  $2a$  has a particle of mass  $m$  attached to each end and it moves in a vertical plane. The midpoint of the rod has coordinates  $(x, y)$ , where the  $x$ -axis is horizontal (within the plane of motion) and  $y$  is the height above a horizontal table. Initially, the rod is vertical, and at time  $t$  later it is inclined at an angle  $\theta$  to the vertical.

Show that the velocity of one particle can be written in the form

$$\begin{pmatrix} \dot{x} + a\dot{\theta} \cos \theta \\ \dot{y} - a\dot{\theta} \sin \theta \end{pmatrix}$$

and that

$$m \begin{pmatrix} \ddot{x} + a\ddot{\theta} \cos \theta - a\dot{\theta}^2 \sin \theta \\ \ddot{y} - a\ddot{\theta} \sin \theta - a\dot{\theta}^2 \cos \theta \end{pmatrix} = -T \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} - mg \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

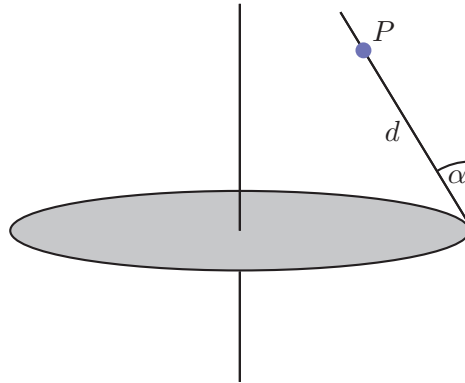
where the dots denote differentiation with respect to time  $t$  and  $T$  is the tension in the rod. Obtain the corresponding equations for the other particle.

Deduce that  $\ddot{x} = 0$ ,  $\ddot{y} = -g$  and  $\ddot{\theta} = 0$ .

Initially, the midpoint of the rod is a height  $h$  above the table, the velocity of the higher particle is  $\begin{pmatrix} u \\ v \end{pmatrix}$ , and the velocity of the lower particle is  $\begin{pmatrix} 0 \\ v \end{pmatrix}$ . Given that the two particles hit the table for the first time simultaneously, when the rod has rotated by  $\frac{1}{2}\pi$ , show that

$$2hu^2 = \pi^2 a^2 g - 2\pi uva.$$

- 11 (i) A horizontal disc of radius  $r$  rotates about a vertical axis through its centre with angular speed  $\omega$ . One end of a light rod is fixed by a smooth hinge to the edge of the disc so that it can rotate freely in a vertical plane through the centre of the disc. A particle  $P$  of mass  $m$  is attached to the rod at a distance  $d$  from the hinge. The rod makes a constant angle  $\alpha$  with the upward vertical, as shown in the diagram, and  $d \sin \alpha < r$ .



By considering moments about the hinge for the (light) rod, show that the force exerted on the rod by  $P$  is parallel to the rod.

Show also that

$$r \cot \alpha = a + d \cos \alpha,$$

where  $a = \frac{g}{\omega^2}$ . State clearly the direction of the force exerted by the hinge on the rod, and find an expression for its magnitude in terms of  $m$ ,  $g$  and  $\alpha$ .

- (ii) The disc and rod rotate as in part (i), but two particles (instead of  $P$ ) are attached to the rod. The masses of the particles are  $m_1$  and  $m_2$  and they are attached to the rod at distances  $d_1$  and  $d_2$  from the hinge, respectively. The rod makes a constant angle  $\beta$  with the upward vertical and  $d_1 \sin \beta < d_2 \sin \beta < r$ . Show that  $\beta$  satisfies an equation of the form

$$r \cot \beta = a + b \cos \beta,$$

where  $b$  should be expressed in terms of  $d_1$ ,  $d_2$ ,  $m_1$  and  $m_2$ .



## Section C: Probability and Statistics

**12** A 6-sided fair die has the numbers 1, 2, 3, 4, 5, 6 on its faces. The die is thrown  $n$  times, the outcome (the number on the top face) of each throw being independent of the outcome of any other throw. The random variable  $S_n$  is the sum of the outcomes.

- (i) The random variable  $R_n$  is the remainder when  $S_n$  is divided by 6. Write down the probability generating function,  $G(x)$ , of  $R_1$  and show that the probability generating function of  $R_2$  is also  $G(x)$ . Use a generating function to find the probability that  $S_n$  is divisible by 6.
- (ii) The random variable  $T_n$  is the remainder when  $S_n$  is divided by 5. Write down the probability generating function,  $G_1(x)$ , of  $T_1$  and show that  $G_2(x)$ , the probability generating function of  $T_2$ , is given by

$$G_2(x) = \frac{1}{36}(x^2 + 7y)$$

where  $y = 1 + x + x^2 + x^3 + x^4$ .

Obtain the probability generating function of  $T_n$  and hence show that the probability that  $S_n$  is divisible by 5 is

$$\frac{1}{5} \left( 1 - \frac{1}{6^n} \right)$$

if  $n$  is not divisible by 5. What is the corresponding probability if  $n$  is divisible by 5?

**13** Each of the two independent random variables  $X$  and  $Y$  is uniformly distributed on the interval  $[0, 1]$ .

- (i) By considering the lines  $x + y = \text{constant}$  in the  $x$ - $y$  plane, find the cumulative distribution function of  $X + Y$ .

Hence show that the probability density function  $f$  of  $(X + Y)^{-1}$  is given by

$$f(t) = \begin{cases} 2t^{-2} - t^{-3} & \text{for } \frac{1}{2} \leq t \leq 1 \\ t^{-3} & \text{for } 1 \leq t < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Evaluate  $E\left(\frac{1}{X + Y}\right)$ .

- (ii) Find the cumulative distribution function of  $Y/X$  and use this result to find the probability density function of  $\frac{X}{X + Y}$ .

Write down  $E\left(\frac{X}{X + Y}\right)$  and verify your result by integration.

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